

RAMAKRISHNA MISSION VIDYAMANDIRA

(Residential Autonomous College affiliated to University of Calcutta)

B.A./B.Sc. FIFTH SEMESTER EXAMINATION, DECEMBER 2024

THIRD YEAR [BATCH 2022-25]

MATHEMATICS [HONOURS]

Date : 12/12/2024

Time : 11.00 am – 1.00 pm

Paper : DSE (Advanced Linear Algebra)

Full Marks : 50

Answer all the questions. Maximum you can obtain is 50.

All the symbols have their usual meaning.

1. If f is a linear functional on \mathbb{R}^3 such that $f(1,0,1)=1, f(0,1,-2)=-1, f(-1,-1,0)=3$ find $f(x,y,z)$. [3]
2. Let W be the subspace of \mathbb{R}^5 spanned by $(1,2,1,0,0), (0,1,3,3,1), (1,4,6,4,1)$. Find a basis for W . [3]
3. Let T be the linear operator on \mathbb{R}^3 defined by $T(x,y,z)=(-9x+4y+4z, -8x+3y+4z, -16x+8y+7z)$. Is T diagonalizable? [6]
4. Let $V = M_n(\mathbb{C})$ be equipped with the inner product $\langle A, B \rangle = \text{Tr}(B^*A)$, $A, B \in V$. Let $M \in M_n(\mathbb{C})$. Define $T : V \rightarrow V$ by $T(A) = MA$. What is $T^*(A)$? [2]
5. Prove that if T^2 has a cyclic vector, then T has a cyclic vector. Is the converse true? [2+3]
6. Prove or disprove : Every linear operator $T : \mathbb{R}^4 \rightarrow \mathbb{R}^4$ has a proper non-trivial invariant subspace. [4]
7. Find the Jordan canonical form and Jordan canonical basis for $A = \begin{bmatrix} 2 & 0 & 0 & 0 \\ -4 & -1 & -4 & 0 \\ 2 & 1 & 3 & 0 \\ -2 & 4 & 9 & 1 \end{bmatrix}$. [15]
8. Let V be \mathbb{R}^2 with standard inner product. Suppose T is a function from V into V such that $\|T\alpha - T\beta\| = \|\alpha - \beta\|$ for all α, β in V and $T(0) = 0$. Prove that T is linear and a unitary operator. [4]
9. Prove that a normal and nilpotent operator is the zero operator. [3]
10. Give an example of two 5×5 nilpotent matrices which have the same minimal polynomials but not similar. [2]
11. Let T be a linear operator on a finite dimensional complex inner product space V . Prove that T is self-adjoint if and only if $\langle Tv, v \rangle \in \mathbb{R}$ for every $v \in V$. [3]
12. Let A be a 4×4 matrix over \mathbb{C} such that $\text{rank}(A) = 2$ and $A^3 = A^2 \neq 0$. Suppose that A is not diagonalizable. What is the characteristics polynomial of A ? Write down the Jordan form of A . [4+2]
13. Suppose T be a normal operator on \mathbb{C}^3 and $T(1,1,1) = (2,2,2)$. Let $(z_1, z_2, z_3) \in N(T)$. Prove that $z_1 + z_2 + z_3 = 0$. [4]